METHOD FOR DETERMINING A POSITION OF AN OBJECT

BACKGROUND OF THE INVENTION

1. Field of the Invention

The invention relates to a method of determining the
position, in particular the future position, of an object, in
5   particular of a particle or the like. The progression of the
position of a reference object is determined, wherein the
reference object passes through a process and differences of
the positions of the object and of the differential object are
formed.

10 2. Description of the Related Art

In many physical and in various technical systems, it
is necessary to know the position, in particular the future
position, of an object, for example in order to avoid damage,
which can occur on account of the fact that the object is located
15 in a position in which such damage is caused. Different methods
to determine the position are known in this context, in which
case the progression of the position of a reference object is
determined and the differences of the positions of the object
and of the reference object are formed. In this context, the
20 reference object is a reference object, whose dynamics can be
determined well. A general problem in the case of the
known methods is that they often yield only inadequate results,

since the methods used for the determination of the position

yield accurate results only at the expenses a very high technical

expenditures. If such technical means are not available, the

results are often unusable.

5   SUMMARY OF THE INVENTION

       Therefore, an object of the present invention is to provide

a method for the determination of the position of an object

of the type mentioned above with the help of which a good

determination of the position of the object is made possible

10   simultaneous with the lowest possible technical expenditures.

       In accordance with the invention, the previously defined

objective is achieved by the method for the determination of

the position of an object with the characteristics of Claim

1. The position may be a future position and the object may

15   be a particle or the like. The claimed method is characterized

by comprising: determining a progression of the position of

a reference object passing through a process; forming

differences of positions of the object and of the reference

object; forming error position terms from the differences; and

20   weighting the error position terms with at least one correction

factor.

       The method at issue for the determination of the position

of an object is designed and modified so that error position

terms are formed from the differences, and that the error

position terms are weighted with at least one correction factor.

It was recognized in accordance with the invention that,

by introducing weighted error position terms, the technical

5   expenditures, for example, the amount of computation during

simulations, can be kept low in a particularly simple fashion.

With respect to a particularly simple embodiment, the

error position terms weighted with the correction factor can

be added up. A correction factor would therefore be obtained

10   that is particularly simple to determine, and also provides

for a good correction. In another advantageous way, the

correction factor could be calculated by an adjoint process.

In this case, the adjoint process could be a process that, as

a function of time, runs in the direction opposite to the process

15   of the reference object. In the context of a particularly simple

embodiment, the weighted and summed error position terms could

be added to the position of the reference object in order to

determine the position of the object.

In the context of a particularly simple embodiment, the

20   object could also be an American option and the reference object

the equivalent European option. The position could then be

the early exercise price and the error position term, in this

case, would be the error price term and the correction factor

the constant.

25   In a manner in accordance with the invention, the early

exercise price could then be calculated for a stipulated

optimization method, for example, the Monte Carlo or

finite-difference method, i.e., the price difference between

the American and European options. Preferably, this could be

achieved by summing of an error price term that is already

5   calculated by most methods. This term could be weighted in

a particularly advantageous fashion by a constant that could

be calculated by a correspondingly selected adjoint process.

The adjoint process, in another advantageous way, could develop,

as a function of time, in the direction opposite to the original

10   process, which determines the price development of the European

options, in which case, for example, a development parameter

could be the formal linking operator of the European price

operator. The price of the American options could then simply

be calculated by estimating the early exercise premium, owing

15   to the fact that said price is added to the price of the

corresponding European option. Numerical errors in the

deficient simulation methods could therefore be compensated

by this method and better estimates achieved than with the

pure numerical methods, on which the method in accordance with

20   the invention is based. For example, this permits a rapid

simulation based on the coarse grids of the finite

difference method. It is thus also possible to overcome the

practical limits of the underlying method and to make the

evaluation of complex options with a number of base values

25   possible.

The method is particularly well suited for estimating

American standard options with a single base value and can be used with particular advantage as a supplement to the finite-difference method and the Monte Carlo method. Implementation occurs in another advantageous way in Java.

5   The method might also be applicable to a variety of controlled or uncontrolled processes, in particular in processes subject to automatic controls. With reference to the already explained embodiment relating to the estimation of option prices, the price of a European option could be an
10   uncontrolled process, since it is independent of external influences of the holder. Accordingly, when determining the price of an American option, a controlled process could be involved, since, in this case, an optimal price determination should take place, i.e., that the option could be exercised
15   as soon as the price falls below the intrinsic value of the option. In this manner, the other embodiments can also be applied and transferred to controlled and uncontrolled processes without difficulty.

There are now various possibilities of advantageously
20   designing and modifying the teachings of the present invention. For this purpose, the claims subordinate to Claim 1, on the one hand, and the subsequent explanation of preferred practical examples of the method in accordance with the invention for the determination of the position of an object with the help
25   of the drawing, on the other, are referred to. Generally preferred embodiments and modifications of the teachings are

also explained in conjunction with the explanation of the

preferred.practical examples of the method in accordance with

the invention for the determination of the position of an object

with the help of the accompanied drawing.

5   BRIEF DESCRIPTION OF THE DRAWINGS

       Fig. 1 shows a practical example of the dependence of

an American option with a risk-free interest rate $r = 0.1$,

volatility $\sigma = 0.4$, exercise price $E = \$10$, in which case the

option expires in six months;

10   Fig. 2 shows the discretization of the prices of the base

values, as well as the allowed transitions for a condition;

and

       Fig. 3 shows another practical example, illustrating the

price of an American put option on a coarse grid by means of

15   a small number of transitions (MC30) and, by adding the adjoint

process correction in the same simulation, the estimated

European option price (APC), with a volatility $\sigma = 0.4$, an

interest rate $r = 0.1$ and an exercise price $E = \$10$.

DESCRIPTION OF THE PREFERRED EMBODIMENTS

American options differ from European options, in that they can be exercised at any time before the expiration of the option, whereas European options can only be exercised on their maturity date. The estimation of American options is

5   therefore much more difficult than the estimation of European options and requires memory- and time-intensive numerical simulations.

It is known that American and European options, under certain conditions, can be equivalent, for example, in the case

10   of an American standard put option on a share, for which no dividend is paid out. Generally, to calculate an appropriate price for an American option, an optimal exercise strategy must be considered at each time step, which makes analytical and numerical calculations much more extensive than in the case

15   of European options.

The estimation of American options is therefore more complex than the estimation of European options. A formula has been developed in this context that permits calculation of an early exercise option price, i.e., the price difference

20   between the two types of options with respect to an adjoint process, which, as a function of time, develops in the backward direction of the original process that determines the development of the price of the European options. This calculation can be used in order to improve option price

25   estimates that are obtained by numerical methods, for example,

the finite-difference or Monte Carlo method.

A method has been developed, which is referred to as the adjoint process correction method (APC), which permits the calculation of the price of an American option by correction

5   of the price of its European equivalent. Its usefulness is demonstrated by application of the APC method to two different numerical methods for an option price calculation, namely, the finite-difference method and stochastic simulation. The latter is based on a reformulation

10 of a development equation with respect to a master equation and is frequently used in physical processes in order to model and simulate a number of both stochastic and deterministic phenomena. Unlike the finite-difference method, the stochastic simulation can be used for practically any known

15 price dynamics.

In this case, the APC method uses both the known backward development equation, and also an adjoint process that runs forward in the direction of time. The price correction can therefore be expressed as an integral over a source term that

20 describes the price discrepancy between the European and the American backward development operators for infinitesimal time steps that are weighted by the adjoint process. The necessary information concerning the adjoint process can often be determined analytically. If this is possible, the APC method

25 can be used particularly simply, with simultaneously low cost, as an expansion of conventional methods. In these cases, the

APC method can precisely develop information that would

otherwise be interfered with by the discretization or

statistical errors and therefore can yield more accurate results

at only slightly higher computational effort than the simple

5   finite-difference method or the Monte Carlo method.

Another advantage of the APC method is that it can be

focused on the relevant price zones. If, for example, the

present price of an option base value is $10, the option price

10   for $20 is not particularly interesting, but, in general, the

finite-difference method will provide this value nevertheless,

just as all other values on a fine grid between $10 to $20,

since the $20 range could be of decisive importance during the

backward calculation of the final conditions. The adjoint

15   process makes it possible to use a comparatively coarse grid

between $10 and $20 and, still obtain very accurate results

for the present price by compression of the initial distribution

of the adjoint process at $10. For options with multiple base

values, the complexity of the problem grows exponentially with

20   the number of base values and therefore a d-dimensional

space must be discretized for d base values. The capability

of the APC method to yield good results, even with coarse grids,

is therefore particularly advantageous in the case

of numerically demanding, higher dimensional cases.

25   In the numerical section of the study, results are

demonstrated for a single base value that follows a geometric

Brownian movement and is processed with the Black-Scholes model in the self-adjoint form. However, no constraints on the APC method per se emerge in this case. For example, the stochastic simulation can be generalized to complicated

5   processes and jump processes can be integrated in a simple fashion. In addition, there is no need for the time development operator to be self-adjoint. A constant time progression is used as starting point, and it is assumed that the development of the European option price vector P is determined by the

10   equation

$$\frac{\partial}{\partial t} P = LP \qquad (1)$$

in which the APC method can be transferred in a simple fashion to discrete time progressions.

Initially, it is demonstrated how the price of a European

15   option can be expressed in the manner of an adjoint process. This result is used and generalized to the case of an American option, which leads to the correction formula that is used in the APC method. In a first numerical application, it is demonstrated that the APC method can be used to calculate option

20   prices by the finite-difference method with a very coarse grid and to interpolate them. An alternative stochastic method is presented that demonstrates that the option price calculation can yield a reasonable price estimate by the adjoint process, when the statistics of the underlying Monte Carlo method yields

25   results that are too poor to furnish reasonable results.

To start with, the price of a European option is developed
in the manner of an adjoint process and definitions and invariant
properties are explained. This process is initially defined
and an obtained quantity found.

5   An adjoint process of process P of equation (1) relates
to a process that is defined by the equation

$$-\frac{\partial}{\partial t}P^{*} = L^{*}P^{*} \qquad (2)$$

in which L^{*} is the formal adjoint of the operator with the standard
L_2 scalar product, so that

10

$$\int_{R^d}\phi(L\psi)\,dx \equiv \int_{R^d}\left(L^{*}\phi\right)\psi\,dx \qquad (3)$$

applies for all ψ, ϕ in the definition range of L or L^{*}. It
is assumed that all ϕ in the definition range of L^{*} are sharply
15   diminishing functions or are vanishingly small for a finite
definition range in the boundary region, in order to avoid
problems in the boundary region during partial integration.
In particular, this is presumed to be the case as a boundary
condition for P^{*}. If, for example, the dynamics is determined
20   by an advection diffusion equation by $L = U\nabla + D\Delta$, the adjoint
operator $L^{*} = -U\nabla + D\Delta$, from which it is apparent that the
process is uniquely defined even in the positive developing
time direction, whereas the adjoint process is uniquely defined
in the backward developing time direction. In the forward
25   developing time direction, the solution to the adjoint process

equation is a poorly solvable task, on account of the presence

of the negative diffusion operator. Accordingly, the temporal

backwards development for the original process is poorly

soluble.

5   With these definitions, the scalar product of P and P*

is time-invariant. Because of the definition of the adjoint

operator in equation (3), for any time points T_1 and T_2 the

following actually applies:

$$\int dx \left(P(x,T_2)P^*(x,T_2) - P(x,T_1)P^*(x,T_1) \right)$$

$$= \int_{T_1}^{T_2} dt \int dx \frac{\partial}{\partial t} \left(P(x,t)P^*(x,t) \right)$$

$$= \int_{T_1}^{T_2} dt \int dx \left(P^*(x,t)\frac{\partial}{\partial t}P(x,t) + P(x,t)\frac{\partial}{\partial t}P^*(x,t) \right)$$

$$= \int_{T_1}^{T_2} dt \int dx \left(P^*(x,t)LP(x,t) - P(x,t)L^*P^*(x,t) \right)$$

$$= 0 \qquad (4)$$

10   The initial or end conditions for the adjoint process can now

be selected in a manner that is most appropriate for the problem

being solved. For this purpose,

$$P^*(x,0) = \delta(x - x_0) \qquad (5)$$

15

is chosen, in which case x_0 is the asset price, at which the

option is to be evaluated, and δ is the d-dimensional Dirac

distribution. Advantageously, x_0 could be introduced as an

additional parameter and P*(x,t; x_0) could be used to designate

20   the solution of the initial value problem of equations (2) and

(5). With these initial conditions, we get

$$P(x_0,0) = \int dx\, P(x,T) P^*(x,T;x_0) \qquad (6)$$

in which T_1 is set to equal 0 and T_2 is set to equal T in equation

5   (4) without adversely affecting the generality of the formula.

Estimation of the option now occurs by means of an adjoint process.
It is presumed that the price of a European option is determined
by the following equation:

10

$$\frac{\partial}{\partial t} P_{eu}(S,t) = L P_{eu}(S,t) \qquad (7).$$

In the model of Black & Scholes and Merton for example, $L =$
L_{BS} is the Black-Scholes operator for a risk-free interest rate

15   r and an underlying volatility σ, which is represented by

$$L_{BS} := -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - rS \frac{\partial}{\partial S} + r \qquad (8).$$

It is apparent from the sign of the diffusion portion

20   $-(1/2)\,\sigma^2 S^2 \partial^2/\partial S^2$ of the operator that the equation is
calculated backward in time.   The end conditions for a put option
on their maturity date T are

$$P_{eu}(S,t) = \max(S - E, 0) =: F(S) \qquad (9)$$

in which E is the base price of the option. With the corresponding
adjoint process $P*$, which is given by

5
$$\frac{\partial}{\partial t} P^{\cdot} = -L^{\cdot} P^{\cdot}$$
$$P^{\cdot}(x,0;S) = \delta(x-S) \qquad (10)$$

and equation (6), we get

$$
\begin{aligned}
P_{eu}(S,0) &= \int dx\, P_{eu}(x,T) P^{\cdot}(x,T;S) \\
&= \int dx\, P^{\cdot}(x,T;S) F(x) \\
&= \int dx\, P^{\cdot}(x,T;S) \max(x-E,0) \qquad (11)
\end{aligned}
$$

10

so that the price of an option can be expressed at the expected
time $t = 0$ and the expected price S in the underlying conditions
of the adjoint process. In the Black-Scholes model, this
corresponds to the option price calculation formula that
15   determines the price as the discounted expectation value of
a profit for a risk-neutral stipulation.

The adjoint process is now applied to the estimation of
American options. For American options, price finding at each
time step entails a decision as to whether the option should
20   be exercised or not. This makes the dynamics of the process more
complicated and different approaches have been derived in order
to model this dynamics, for example, as a free-boundary,
linear-complementarity or optimal-stopping problem. It is

assumed in this case that, for infinitesimal time steps dt, the time development of the option price is represented by the following equation:

5      $$P_{am}(x, t + dt) = \max(P_{am}(x, t) + LP_{am}(x, t)dt, F(x, t + dt)) \qquad (12)$$

in which case $F(x, t)$ is a payoff function that reflects the intrinsic value of the option, which depends on both the present status x of the base value, and also of time t. This formulation

10   can be derived, for example, from the Black-Scholes model, as an explicit discretization of the corresponding linear complementary problem, as demonstrated by Wilmott, Howison and Dewynne. Similar equations can also be derived from discrete time formulations, for example, the dynamic programming

15   equation for binomial trees.

   Considering

$$\max(P_{am} + LP_{am}dt, F) = P_{am} + LP_{am}dt + \max(F - P_{am} - LP_{am}dt, 0) \qquad (13)$$

20   the equation for $dt \to 0$ can be rewritten

$$\frac{\partial}{\partial t} P_{am} = LP_{am} + J(F, P_{am}, t) \qquad (14)$$

with a source term

25

$$J(F, P_{am}, t) := \lim_{dt \to 0} \frac{\max(F - P_{am} - LP_{am}dt, 0)}{dt} \qquad (15)$$

In the equation (15), a differential object may be defined as an object that at the previous time step had the same

5   state/position as the object but undergoes the same (simple) dynamics as the reference object in the next step. In the case of numerical solution of an American option, this differential object would correspond to an option priced with $P_{am} + LP_{am}dt$ after a time step with size dt.

10   It should be noted that the source term does not vanish in the region of exercise only. In the region of exercise, in each case, $P_{am}(x, t) = F(x, t)$, so that the following is obtained:

$$15 \qquad J(F, P_{am}, t) := \begin{cases} \dfrac{\partial}{\partial t} F - LF & : x \text{ in exercise region} \\ 0 & : x \text{ outside of exercise region} \end{cases} \qquad (16)$$

It can also be determined without numerical simulations from equations (15) and (16) whether the American option differs from the European option. This would be possible, in particular,

20   if it applies that $(F - P_{am} - LP_{am}dt)$ and/or $(\partial / \partial t\, F - LF)$ ≤ 0. In this case, the source term $J(F, P_{am}, t)$ disappears, from which it follows that the two systems behave essentially identically. A practical example for this case is an American standard put option for shares without dividends in the Black

Scholes model.

It is particularly advantageous in this case that numerical efforts can be avoided, since the more easily determined price of the European option could now be determined. In addition, products could also be deliberately with this method that behave essentially like a European or American option. By selecting the payoff function F, the behavior of the option can be influenced, so that $J(F, P_{am}, t)$ becomes non-negative or non-positive. A premature exercise possibility could then be advantageous for both sides, in particular if it is known to the offering party whether an option can be offered without or with only limited risk premium as an American option.

The adjoint process $P*$ is now defined as in equation (2) with L of equation (12), i.e., unlike P_{am}, the adjoint process is determined as in the case of European dynamics. If the calculation of the European options is repeated and it is assumed that the present time is zero and the exercise time T, then, without restriction of the generality, the following applies:

$$
\int dx \left(P_{am}(x,T)P^*(x,T) - P_{am}(x,0)P^*(x,0) \right)
$$

$$
= \int_0^T dt \int dx \frac{\partial}{\partial t} \left(P_{am}(x,t)P^*(x,t) \right)
$$

$$
= \int_0^T dt \int dx \left(P^*(x,t)\frac{\partial}{\partial t} P_{am}(x,t) + P_{am}(x,t)\frac{\partial}{\partial t} P^*(x,t) \right)
$$

$$
= \int_0^T dt \int dx \left(P^*(x,t)LP_{am}(x,t) - P_{am}(x,t)L^*P^*(x,t) + J(F,P,t)P^*(x,t) \right)
$$

$$
= 0 + \int_0^T dt \int dx\, J(F,P_{am},t)P^*(x,t) \qquad (17)
$$

It is known from equation (11) that the integral $\int dx\, P_{am}(x, T)\, P^*(x, T) = \int dx\, F(x, t)\, P^*(x, T)$ in equation (17) is equal to the European option price P_{eu}. If $P^*(x, 0) = \delta(x-S)$ now applies as the initial condition of the adjoint process, and we obtain

5

$$P_{am}(S,0) = P_{eu}(S,0) + \int_0^T dt \int dx\, J(F, P_{am}, t) P^*(x,t;S) \qquad (18)$$

from equation (17), so that the early exercise option price is expressed in the form of an integral over the incorrect price

10   source term J, which is weighted by the adjoint process P^*.

Using equation (16), equation (18) can then be further simplified to

$$P_{am} = P_{eu} + \int_0^T dt \int_{\Omega(t)} dx\, P^*(x,t)\left(\frac{\partial}{\partial t} - L\right) F(x,t) \qquad (19)$$

15

in which case $\Omega(t)$ is the region of exercise at time t.

In this context, equation (16) is a generalization of the integral equations derived by Kim, Jacka, Carr, Jarrow and Myeni for the Black-Scholes model with a single base value.

20   This formula could be used, for example, as illustrated later, for semi-analytical methods, in order to find an exercise limit. It applies in this case that, in the Black-Scholes model, equation (19) under certain conditions yields the decomposition formula that is derived in these studies.

The numerical application of the APC method in the finite-difference method is explained below. In this case, the decomposition in equations (18) and (19) is seen as a tool that improves the numerical methods for solving the problem

5   of the estimation of American options.

There are three properties that make this equation interesting for the numerical determination of option prices:

1. The source term is different from zero only in the region of the exercise. In this region, the exact solution

10   is also known since it equals the payoff function F. The source term can therefore be calculated with a high degree of accuracy either numerically or analytically.

2. When the adjoint process can be analytically determined, the contribution of these sources to the option

15   price correction can be propagated with a precise transfer function $P^*(S, T; S_o)$, to S_o.

3. In the case of the Monte Carlo method, the double integral smoothes fluctuations in local price estimates.

The equations (18) and (19) can also be used to improve

20   the price estimate of a simple explicit finite-difference method. To explain this, a Black-Scholes operator of equation (8) is used. The use of

$$x = \ln \frac{S}{E}$$

$$\tau = \frac{1}{2}\sigma^2(T-t)$$

$$P = Ee^{\alpha x + \beta t} p(x, \tau) \qquad (20)$$

with $\alpha = (1/2)(k - 1)$, $\beta = -(1/4)(k + 1)^2$, $k = 2r / \sigma^2$ transforms
the Black-Scholes equation into a self-adjoint form

5

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} \qquad (21)$$

that is very much easier to solve numerically. The payoff
function $F(S) = max(E - S, 0)$ is transformed into the following

10   equation

$$f(x, \tau) = e^{\frac{1}{2}(k+1)^2 \tau} \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0 \right) \qquad (22)$$

and the initial conditions are

15

$$p(x, 0) = f(x, 0) = \max\left(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0 \right) \qquad (23)$$

The boundary conditions can now be derived as follows:

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$$P(0,t) = Ee^{-r(T-t)} \qquad (24)$$
$$P(S,t) \to 0 \quad for \ S \to \infty \qquad (25)$$

An explicit finite-difference discretization of equation (21) with a normal grid $\{x_0, \ x_1, \ x_2 \dots x_N\}$, $x_n = x_0 + n\sigma x$ and with a constant time step δt is used in order to explain the APC method in the finite difference scheme. With the help of the definition $p_n^m = p(x_n, \ m\delta t)$, the discretization is obtained for a European put option from

$$\frac{p_n^{m+1} - p_n^m}{\delta t} + O(\delta t) = \frac{p_{n+1}^m - 2p_n^m + p_{n-1}^m}{(\delta x)^2} + O\big((\delta x)^2\big) \qquad (26)$$

If the terms of order $O(\delta t)$ and $O((\delta t)^2)$ are ignored, the following difference equation is obtained:

$$p_n^{m+1} = \alpha p_{n+1}^m + (1 - 2\alpha)p_n^m + \alpha p_{n-1}^m \qquad (27)$$

in which

$$\alpha = \frac{\delta t}{(\delta x)^2} \qquad (28)$$

It is known in this case that this update method is stable for $0 < \alpha \leq 1/2$. For an American option, equation (27) must be replaced by the following equation

$$p_n^{m+1} = \max\left(\alpha p_{n+1}^m + (1-2\alpha)p_n^m + \alpha p_{n-1}^m, f_n^{m+1}\right) \quad (29)$$

in which $f_n^m = f(x_0 + n\delta x, m\delta t)$ is the discretized payoff function.

5   The initial and boundary equations are determined by equations (23) and (25).

Based on adjoint process corrections in the finite -difference scheme, equation (16) is used to calculate the early exercise price as an addition to the

10   finite-difference method. Equation (21) is present in the self-adjoint form, so that

$$L^* = L = \frac{\partial^2}{\partial x^2} \quad (30).$$

15   The solution of the adjoint problem defined in equation (10) is

$$p^*(x,\tau;x_0) = e^{-(x-x_0)^2/(2\tau)} \quad (31)$$

20   in which case $x_0 := \ln(S_0/E)$ and S_0 is the price of the base value to be determined, i.e., in most cases, the present price.

The exercise price $P_{am} - P_{eu}$ could be calculated by summing of $\max(F - P_{am} - \mathcal{L}P_{am}(dt, 0))$ at each grid point and at each time step.

25   Fig. 1 shows the dependence of American option at a

risk-free interest rate $r = 0.1$, volatility $\sigma = 0.4$ and the exercise price $E = \$10$, in which case the option expires in six months and was calculated on a coarse grid with the explicit finite-difference method (50 grid points). Fig. 1 also shows,

5   on the same grid, the dependence of an American option by means of the adjoint process correction term, which is added to the European price. (50 grid points, APC method), and Fig. 1 also shows a reference solution that is calculated by a Crank-Nicolson scheme on a fine grid, and the corresponding price of the European

10   option.

The results for a simulation on a coarse grid are shown in Fig. 1. Significantly better estimates of the American option price are obviously obtained, if the price is calculated by adding the early exercise price, which is calculated with

15   equation (18), to the European option price, than in the case of the direct calculation with the finite-difference method, although, in both methods, the same grid is used and the summation of the weighted source terms requires only a limited amount of computation. It is also obvious that the linked process

20   correction yields a smooth estimate over the entire interval, whereas the finite-difference method uses an interpolation method between grid points.

In order to demonstrate the use of equation (15) for Monte Carlo methods, i.e., the estimation of American options with

25   the help of the stochastic simulation, it is applied to the stochastic simulation of a master equation. This is a numerical

approach that has numerous applications in physics in order
to solve a master equation that describes a time development
of a probability P_n of finding the system in state n:

5      $$\frac{\partial P_n}{\partial t} = \sum_m R_{nm} P_m - \sum_m R_{mn} P_n \qquad (32)$$

in which case R_{mn} is the transition rate for the transition
of the system from state m to state n. In this case, equation
(32) must be interpreted as a system of ordinary differential
10   equations for all states.

        The use of the adjoint process correction in a
Black-Scholes model is shown again. In every case, this
approach is not limited to this model and effects, like the
stochastic volatility and jump processes, can also be integrated
15   in a simple fashion into the master equation.

        Despite its flexibility and simplicity, the stochastic
simulation is not a standard method that is used in
finance mathematics. Faller and Petruccione have presented
a master equation approach for the option price estimation,
20   which, however, differs in important points from the method
shown here, for example, since they use constant time steps.
The stochastic simulation approach of Faller and Petruccione
is more rapid than the standard Monte Carlo methods, but not
as effective as the approach illustrated here.

25   The use of the minimal process method will now be explained.

The idea behind the minimal process method is to establish a simple Markov process, whose state probabilities obey the master equation (32). When a representation of the state is found, the algorithm works as follows:

5   1. Selection of an initial state n according to the initial conditions.

          2. Adding up of the transition rates for all states m that can be achieved from state n in a single transition: R_{tot} $= \Sigma_m R_{mn}$

10   3. Advance in time by a time increment τ that is obtained from an exponential distribution and has the mean R_{tot}, i.e., with the probability density $\pi(\tau) = R_{tot}^{-1} \exp(-R_{tot}\tau)$

          4. Selection of a new state m from all possible states with the probability R_{mn}/R_{tot}

15   5. Performance of the transition to the selected state m, i.e., n is set equal to m.

          6. Go to step 2, until the increasing time is greater than or equal to the desired simulation time.

20   The most time consuming part of this algorithm is step 4, the selection of the actual transition, since the number of end states that can be reached from m can be very large. In the following example, it is $O(N)$, in which case N corresponds to the number of used grid points. A linear selection algorithm would therefore also be $O(N)$. In this case, in order to accelerate the selection process, the "dual logarithmic classes

approach" of Fricke and Schnakenberg is used, in which all

possible transitions are organized in reaction categories. The

transition rate varies within a category at most by a factor

of 2, so that transitions are selected very efficiently by a

5   von Neumann evaluation scheme, once the reaction category is

chosen. The number of reaction categories is small in this

case (the number of octaves that is spanned by the minimal

and maximal transition rate), so that only a few steps are

required in order to select a reaction category with the

10   probability R_c/R_{tot} by a linear selection, in which case R_c is

the sum of all transition rates in category c.

In the interest of simplicity – as a discretization of

the Black-Scholes equation by a master equation – the diffusion

equation can be used again below, i.e., the Black-Scholes

15   equation transformed into a self-adjoint form.

The process is discretized under the following

assumptions:

1. An equidistant discretization of the logarithm of

the price of the base value is conducted from $x_0 = x_{min}$ to x_N

20     $= x_{max}$, with $x_i - x_{i-1} = \Delta x$ for $i = 1 \ldots N$.

2. The states for each space $[x_i, x_{i-1})$ are constant step

functions – as shown in Fig. 2, which shows the discretization

of the price of the base value and the permitted transition

for a state.

25   3. For each space, a jump of a quasi particle to its next

left or right neighbor is permitted. The particle carries a

fixed, defined fraction Δp of the option price, so that, after

one jump from *i* to *i −1*, the actualized prices are *p(i, t) =*

p(i, t − τ) − Δp and *p(i − 1, t) = p(i − 1, t − τ) + Δp* for

the European option and *p(i, t) = max(p(i, t − τ) − Δp, f(i,*

5   *t)* and *p(i − 1, t) = max(p(i − 1, t − τ) + Δp, f(i, t))* for

an American option.

        This obviously describes a random, one-dimensional path

for each quasi-particle. Accordingly, the probability density

of a group of quasi-particles is subject to a diffusion equation

10   with the limit Δx → 0. In the self-adjoint form of the equation

(21), the diffusion coefficient equals one, so that the

transition rate for a step process of a quasi-particle from

i to *i ± 1* is set, independently of *i*, at

$$r = 1 / \Delta x^2 \qquad (33)$$

15   The efficiency of the stochastic simulation could also

be improved further by means of the introduction of net

transition rates. The number of quasi-particles in space *i*

at time *t* is *N_i = p(i, t) / Δp*. Therefore, from averaging of

all transitions that begin at the present time, it is apparent

20   that the net transfer of quasi-particles from node *i* to *i ±*

1 is obtained from

$$j_i = \begin{cases} r(N_i - N_{i-1})\langle \tau \rangle & : N_i > N_{i\pm 1} \\ 0 & : N_i \leq N_{i\pm 1} \end{cases} \qquad (34)$$

25   in which ⟨τ⟩ is the average size of the time step and this is

considered by the fact that the net transition rate for each

jump is introduced to each space.

$$r(i|i \pm 1) = \begin{cases} r(N_i - N_{i-1}) & : N_i > N_{i\pm1} \\ 0 & : N_i \leq N_{i\pm1} \end{cases} \quad (35).$$

5

Using these net transition rates has two special advantages.

In the first place, the movements of the occupation numbers

in the spaces, i.e., in the price estimates, are sharply reduced.

On the other hand, the sum over the net transition rates is

10   far smaller than the sum over the transition rates of the

quasi-particles. Since the typical time step is inversely

proportional to the total transition rate, this means that the

simulation is ended much more quickly.

In Fig. 3, the price of an American put option is calculated

15   on a coarse grid by means of a small number of transitions (MC30)

and the European option price (APC) is estimated by adding the

adjoint process correction in the same simulation. The

volatility in this example is $\sigma = 0.4$, the interest rate $r =$

0.1 and the exercise price $E = \$10$.

20   As an example of a numerical experiment, Fig. 3 shows

the results of an estimate for an American option, in which

case a very coarse grid with 30 grid points is used and 600

transitions are carried out in a single implementation of the

stochastic process. The option prices that are estimated by

25   a backward simulation are practically without value: in many

grid points, the predicted American price lies below the European

price, which is impossible. The APC method is based on the

same simulation and yields, by contrast, with only a slightly

increased computational effort, very good estimates of the

5   actual American option price.

Finally, and to summarize, it can be stated that the early

exercise price for an American option can be represented by

a compact formula, in which use is made of an adjoint process

that is developed in the backward direction of time of the

10   original process, in which case the development of the

corresponding European option price is determined. This

formula, in an inventive manner, generalizes an integral

equation that is valid in the Black-Scholes model and that is

used in the literature to derive semi-analytical methods for

15   the estimation of American options.

It is demonstrated that the formula can be used as an

addition for numerical standard methods in order to obtain

decidedly better estimates of the early exercise price. This

is demonstrated on the example of the Black-Scholes model for

20   a finite-difference method. In order to demonstrate use of

the formula for Monte Carlo methods, a stochastic simulation

based on a master equation representation of the process is

introduced. A sufficiently good estimate of the early exercise

price can be obtained even for very coarse approximations with

25   poor statistics. The reason for the good performance is that

the formula permits an analytical or semi-analytical

propagation of the source term generated in the exercise region

to be conducted.   Consequently, the greatest cause for numerical

errors lies in uncertainty of the location of the exercise limit.

Obviously, the adjoint process can also be used to

5   develop new semi-analytical and numerical methods. If, for

example, one has an initial estimate of the exercise limit,

the stochastic simulation method can be altered in such a manner,

that one can operate in the exercise region without

discretization, since the prices, and therefore the transition

10   rates, are known there analytically. In the following areas,

the values can be calculated by summing the adjoint process

corrections over the exercise region, so that, in the case of

a self-consistent calculation, a very coarse discretization

about the limit is sufficient. In combination with the fact

15   that estimates obtained by an adjoint process are usable even

with very coarse grids, it follows that the method can also

be used very well for estimates of options with several base

values.

Finally, the adjoint process offers a particularly good

20   possibility of interpolating and extrapolating prices that are

also calculated with coarse grids.

The following explicit breakdown of the price of an

American put option into the European price and the early

exercise price in the Black-Scholes model is known in the

25   literature:

$$P_{am}^{BS}(S_0,0) = P_{eu}^{BS}(S_0,0) + rE \int_0^T e^{-rt} N\left(-\frac{\ln B_t/S_0 + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}} \right) dt \qquad (36)$$

in which case B_t is the location of the exercise limit at time

t and

5

$$N(x) := (2\pi)^{-1/2} \int_{-\infty}^x e^{-\xi^2} d\xi \qquad (37)$$

applies.

This is a special case of equation (19), since, when

10   $L = L_{BS}$, the adjunct operator from equation (8) is obtained from

$$L_{BS}^* = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \left(r - \frac{3}{2}\sigma^2\right)S\frac{\partial}{\partial S} + 2r - \sigma^2 \qquad (38)$$

as can be demonstrated by the partial integration in the

15   determining equation of the adjoint operator; see equation (3).

The adjoint operator, in this case, is not identical to the

Black-Scholes operator, but has a similar structure.   The

adjoint problem

20   $$\frac{\partial}{\partial t}P^* = -L_{BS}^* P^*$$
$$P^*(S,0;S_0) = \delta(S - S_0) \qquad (39)$$

can be solved analytically by the same methods used to derive

the Black-Scholes formula, in which case significantly simpler

initial conditions can be used. The result is

5 $$P^*(S,t;S_0) = S_0^{-1} \exp\left(-\frac{\left(\ln S - \ln S_0 - \left(r - 3/2\sigma^2\right)t\right)^2}{2\sigma^2 t}\right) \frac{e^{-\left(2r-\sigma^2\right)t}}{\sqrt{2\pi\sigma^2 t}} \qquad (40)$$

The payoff function of an American put option is

$$F(S,t) = F(S) = \max(E - S, 0) \qquad (41)$$

10

so that, within the exercise region, the following applies

$$\left(\frac{\partial}{\partial t} - L_{BS}\right) F(S,t) = rE \qquad (42)$$

15   The exercise region, in this case, is the interval $\Omega(t)$
= $[O,\ B_t]$ and it follows from equation (19) that

$$P_{am}^{BS}(S_0,0) = P_{eu}^{BS}(S_0,0) + \int_0^T dt \int_0^{B_t} dS \frac{rE}{S_0} \exp\left(-\frac{\left(\ln S - \ln S_0 - \left(r - 3/2\sigma^2\right)t\right)^2}{2\sigma^2 t}\right) \frac{e^{-\left(2r-\sigma^2\right)t}}{\sqrt{2\pi\sigma^2 t}} \qquad (43)$$

20 By the substitution $\zeta = \ln S/S_0$, and the quadratic expansion

in the resulting exponential argument, this can be brought into

the form of equation (36).

With respect to the additional advantageous embodiments

and modifications of the method in accordance with the invention, the general description and the enclosed claims are referred to in order to avoid repetitions.

Finally, it must be expressly pointed out that the
5   practical examples described above serve only to explain the claimed teachings, but that said teachings are not restrict to the practical examples.